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DIFFUSION POTENTIAL AND WELL-POSEDNESS IN NON-ASSOCIATIVE PLASTICITY

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Abstract—The problem of ill-posedness of the boundary value problem in non-associative plasticity is addressed. Well-posedness is achieved through the introduction of a potential ϕ that plays the dual role of a hardening function and a diffusion potential. Simultaneous constitutive equations in the form of partial differential equations for the velocity field $\dot{u}_i$ and ϕ are established. The problem of simple shearing is solved as an example and patterned deformation is shown to occur in the presence of softening. The work is confined to small deformation. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Non-associative theories of plasticity arise when the plastic potential surface (to which the plastic strain rate is normal), does not coincide with the yield surface. Such is the situation when the position and configuration of the yield surface in stress space, depend on the prevailing hydrostatic stress. Correct prediction of the volumetric plastic strain rate then calls for non-associative theories.

However, when non-associativity prevails, the boundary value as well as the wave propagation problem are not unconditionally well-posed, even in the absence of softening, as is the case in associative theories. See Valanis and Peters (1996), where it is demonstrated that well-posedness fails in the case of a plastically incompressible solid with a pressure-dependent yield surface. In a much earlier paper Rudnicki and Rice (1975) showed that the determinant of the acoustic tensor $D(\underset{\sim}{Q}_n)$ vanishes in the case of soil model with a pressure dependent yield surface, thus demonstrating the existence of shear bands. The condition $D(\underset{\sim}{Q}_n) = 0$, is tantamount to loss of hyperbolicity and therefore, ill-posedness. The difficulties cannot be addressed in the context of a local theory.

In cases where ill-posedness is due to softening, "regularization procedures" involving higher gradients of appropriate constitutive variables have been proposed in recent years, but it is not always clear how these reflect the underlying physics of the problem. Also the associated boundary conditions are often physically opaque and/or difficult to replicate experimentally.

A leading approach in Mises plasticity, used by Aifantis (1992a), in the presence of plastic incompressibility, is to introduce in the hardening function of the yield law, a dependence on the Laplacian of the intrinsic time z, as in eqn (1.1):

$$f(\underset{\sim}{s}) = k(z, \nabla^2 z) \tag{1.1}$$

where $\underset{\sim}{s}$ is the deviatoric stress tensor, $f(\cdot)$ is the yield function and $k(\cdot)$ the hardening function, which in standard plasticity depends on z alone.

In this event the Prager consistency condition, i.e., the rate form of eqn (1.1), in the material domain D, becomes:

$$\underset{\sim}{f}\cdot\dot{\underset{\sim}{s}} = k_z\dot{z} + k_\nabla \nabla^2 \dot{z} \tag{1.2}$$

where $\underset{\sim}{f} = \partial f/\partial \underset{\sim}{s}$, $k_z = \partial k/\partial z$ and $k_\nabla = \partial k/\partial \nabla^2 z$. Equation (1.2) is a linear second order partial differential equation in $\dot{z}$ with a "forcing function" on the left hand side. If a solution of this equation is to exist and be unique, either $\dot{z}$ or $\partial\dot{z}/\partial n$ must be known on the boundary S of D, where $\partial\dot{z}/\partial n$, i.e., $\nabla\dot{z}\cdot n$, is the projection of the gradient of $\dot{z}$ onto the unit normal to the surface. In this regard see Muhlhaus and Aifantis (1991).

Difficulties arise because neither $\dot{z}$ or $\partial\dot{z}/\partial n$ is known on the boundary, nor is the manner in which experimental conditions affect these quantities on the boundary, understood. Thus unless means can be found for resolving these questions, these issues will remain.

Despite these difficulties, a great deal of pioneering work has been done in this area by Aifantis (1992b), who must be credited for pointing research efforts in the right direction to understand the formation of bands, and to resolve the question of ill-posedness in the presence of softening. For other contributions in this area see Vardoulakis and Aifantis (1991) and Vardoulakis and Frantziskonis (1992). More recently, Valanis (1996) developed a gradient theory of internal variables and showed, Valanis (1998), that it leads to banding and patterned deformation in solids.

In this paper we begin with the observation that internal motion in materials is not affine and is in fact diffusive. Dislocation motion makes the case in metals. For granular media it was demonstrated experimentally by Bridgewater *et al.* (1985). Our point of departure then, is that a component of the volumetric deformation is attributable to changes in a diffusion potential akin to the configurational temperature of the material, as discussed previously by Valanis (1993).

The resulting non-local constitutive framework preserves the physics of the local theory, which does well in homogeneous deformation, and involves boundary conditions that are physically motivated and experimentally realizable. More importantly, it leads to boundary value problems that are well-posed.

We shall limit the analysis to materials that are plastically incompressible. This is very closely true of metals at room temperature and approximately true in soils in situations where the shear strain far exceeds its volumetric counterpart. In metals one encounters the strange phenomenon where the yield surface is susceptible to changes in pressure and yet the plastic volumetric strain remains virtually negligible. See Richmond and Spitzig (1980).

The resulting theory of such behavior must be non-associative since normality of the plastic strain rate to the yield surface would lead to excessive plastic volumetric strain. It was shown however, Valanis and Peters (1996), that such a theory leads to ill-posed initial and boundary value problems.

Summary of the paper

In Section 1 we give a brief review of regularization to the problem of softening.

In Section 2 we present a short discussion (also done elsewhere by other authors) of a plastically incompressible Mises solid with a pressure dependent yield surface. This solid is non-associative, leading to violation of Drucker's stability postulate and lack of unconditional well-posedness of the boundary value problem. The discussion is limited to isotropic hardening. An analysis in the presence of isotropic as well as kinematic hardening is given in Section 9.

In Section 3 we introduce a void flux q and a potential ϕ, the mathematical purpose of which is to eliminate ill-posedness in the presence of non-associativity. The potential ϕ plays a dual physical role. The first is a hardening role (as in Hertzian friction) very much like the *pressure* in the local theory, and the second is the role of a diffusion potential, such that q is linearly related to the gradient of the rate of ϕ, i.e., $\nabla\dot{\phi}$.

A constraint, crucial to the role of $\dot{\phi}$, is that the rate of the void volume generated *in situ* by $\dot{\phi}$, plus that generated by diffusion, must sum to zero, to preserve plastic incompressibility. The law of evolution for $\dot{\phi}$ is given by eqn (3.11).

In Section 4 we give the complete constitutive formulation which relates (i) the stress rate $\dot{\underset{\sim}{\sigma}}$ to the strain rate $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{\phi}$ and (ii) the partial differential equation that relates $\dot{\phi}$ to $\dot{\underset{\sim}{\varepsilon}}$. These relations are given by eqns (4.9) and (4.10), respectively.

In Section 5 the resulting well-posedness of boundary value problem is then discussed with appropriate physically realizable boundary conditions motivated by the physics on the boundary.

In Section 6 the workings of the theory are illustrated by solving the problem of a material layer of finite thickness but infinite extent, subjected to simple shear on the upper boundary but bonded to an impermeable foundation. When hardening prevails the problem is solved in closed form and the solution is seen to be unique and stable. The non-local character of the solution is discussed and its differences from the local solution are pointed out.

In Section 7 we deal with the question of a softening material by finding a solution, again in closed form, in the presence of softening. This solution is important in three respects:

(i) It exists and is unique.
(ii) It is stable except at discrete, singular points of instability.
(iii) It is periodic, giving rise to patterned deformation fields, as observed.

Thus, the potential ϕ rectifies ill-posedness due to non-associativity as well as material softening.

In Section 8 we show that the variational solution to the boundary value problem exists and is unique, in the light of an inequality that governs the material constants pertaining to the diffusion processes in the solid.

In Section 9 we give the full constitutive formulation in the presence of both isotropic and kinematic hardening.

2. LOCAL THEORY—PLASTIC INCOMPRESSIBILITY IN THE PRESENCE OF A PRESSURE-DEPENDENT YIELD SURFACE

Solid with isotropic hardening

To fix ideas we begin with the simple case of a plastically incompressible solid with isotropic hardening and pressure-dependent yield surface as depicted in eqn (2.1):

$$f(s_{ij}) = k(p, z) \tag{2.1}$$

where $\underset{\sim}{s}$ is the deviatoric part of the stress tensor $\underset{\sim}{\sigma}$, $p = -\sigma_{kk}/3$ and z is the intrinsic time scale. We remark that k satisfies the conditions: $k_z > 0$, $k_p > 0$, where $k_z = \partial k/\partial z$ and $k_p = \partial k/\partial p$. Quite clearly, one cannot admit an associative theory in the light of eqn (2.1), since then a plastic volumetric strain would exist, whose rate would be equal to $-\dot{z}\,\partial k/\partial p$. Thus, a non-associative theory must be adopted where f acts as a plastic potential and the plastic deviatoric strain rate $\dot{\underset{\sim}{e}}$ is given by eqn (2.2a,b):

$$\dot{\underset{\sim}{e}}^p = \dot{z}\underset{\sim}{f}, \dot{z} > 0, \quad \dot{\underset{\sim}{e}}^p = 0, \quad \dot{z} \leqslant 0 \tag{2.2a,b}$$

where $\underset{\sim}{f} = \partial f/\partial \underset{\sim}{s}$, and $\dot{\varepsilon}^p_{kk} \equiv 0$. It then follows that:

$$\dot{\underset{\sim}{e}}^e = \dot{\underset{\sim}{s}}/2\mu; \quad \dot{\varepsilon}_{kk} = \dot{\varepsilon}^e_{kk} = \dot{\sigma}_{kk}/3K \tag{2.3a,b}$$

where μ and K are the shear and bulk elastic moduli, respectively. Also,

$$\dot{\underset{\sim}{e}} = \dot{\underset{\sim}{e}}^e + \dot{\underset{\sim}{e}}^p \tag{2.4}$$

The Prager consistency condition, i.e., eqn (2.1) in rate form is given by eqn (2.5):

$$f_{ij}\dot{s}_{ij} = k_p\dot{p} + k_z\dot{z} \tag{2.5}$$

or:

$$k_z\dot{z} = f_{ij}\dot{s}_{ij} - k_p\dot{p} \tag{2.6}$$

We also note the relation:

$$\dot{e}^p_{ij}\dot{s}_{ij} = \dot{z}f_{ij}\dot{s}_{ij} \tag{2.7}$$

which follows from eqn (2.2a).

Remark. Note that f_{ij} is normal to the yield surface. Thus, if $\dot{p} = 0$, then $\|\dot{\underset{\sim}{e}}^p\| \neq 0$. i.e., $\dot{z} > 0$ if the angle α between $\underset{\sim}{f}$ and $\dot{\underset{\sim}{s}}$ is acute. However, when $\dot{p} < 0$ it is possible for $\|\dot{\underset{\sim}{e}}^p\| \neq 0$ even when the angle $\alpha > \pi/2$ ($f_{ij}\dot{s}_{ij} < 0$), provided that $k_p|\dot{p}| > |f_{ij}\dot{s}_{ij}|$. In view of eqn (2.7), the inequality $f_{ij}s_{ij} < 0$ means that:

$$\dot{e}^p_{ij}\dot{s}_{ij} < 0 \tag{2.8}$$

When eqn (2.8) applies, the stability condition ($\dot{e}^p_{ij}\dot{s}_{ij} > 0$) as proposed by Drucker (1959) is violated and the boundary value problem is no longer well posed, since given a domain D with surface S and traction rates $\dot{T}_i$ on S, the problem may either have no solution or admit a multiplicity of solutions for the strain field in D. See Valanis and Peters (1996).

3. THE DIFFUSION POTENTIAL

Deformation processes are usually accompanied by diffusion of dislocations in the case of crystals, particle (or void) diffusion, or coordinated dislocation diffusion such as intra- and/or inter-granular slip in the case of metals and migration of granules (or voids) in the case of granular soils. See Bridgewater *et al.* (1985). Close observation of the motion of granular media, reveals that the applied loads (stresses) are carried by a solid "skeleton" formed by granules within the material domain, and to a lesser extent by looser grains that engage in migratory motion. The skeleton does not consist of the same grains, but undergoes a reformation process during the deformation history.

Thus, a material domain consists of a solid and a migratory phase, and particles or grains enter and leave a phase in the course of deformation. There is no evidence of conservation of number of particles or grains, in either phase, during a stress history. This last mode of deformation is not encountered in local theories of plasticity, such as the one presented in Section 2.

Moreover, because diffusive motion calls for a potential gradient leading to a gradient theory, we need to reinterpret experimental observation. We note that experiments, at the phenomenological level, are carried out on finite regions and within these there exists migratory motion resulting in inhomogeneous stress and strain fields in cases where local theories would predict otherwise.

Thus, while we say that a domain is subjected to a hydrostatic stress, we mean that a finite region at its boundary, is subjected to traction that are normal to the boundary. Also when it is said that the strength of a neighborhood increases with increasing pressure, what is meant is that normal compressive traction increase the overall shear strength of the domain. Since, in general, the stress distribution within the domain is not known, unless a local theory is assumed, we cannot say a great deal about the interior hydrostatic stress or other "strength-increasing" interior stresses.

In the light of these remarks we introduce a new constitutive variable ϕ and we reinterpret eqn (2.1) to mean that:

$$f(\underset{\sim}{s}) = k(\phi, z) \tag{3.1}$$

ϕ being such that on the permeable part of the boundary where tractions T_i are applied:

$$\phi = -T_i n_i \tag{3.2}$$

where n_i is the unit normal to the surface. The sign convention in eqn (3.2) means that ϕ on the surface is positive when the normal boundary traction are compressive.

Let uniform tractions be applied to the permeable faces of a cubical domain. In local theory parlance, the normal traction on face one, normal to x_1, is σ_{11} that on face two, normal to x_2 is σ_{22} and that on face three, is σ_{33}. Thus, in the light of eqns (3.1) and (3.2), the shear strength (and stress) in a neighborhood adjacent to face "one" will be influenced by σ_{11}, that of a face "two" by σ_{22} and so on. This situation is much more depictive of a theory of Hertzian friction than a theory where the shear strength of the cube is dependent on the hydrostatic stress $(1/3)\sigma_{kk}$.

The role of the potential ϕ is two-fold: (i) $\dot{\phi}$ itself causes void generation *in situ* by local structural material changes leading to void accretion or annihilation, while (ii) the gradient of $\dot{\phi}$ causes void diffusion to (or from) neighboring material elements. We thus let ω_ϕ be the total void density generated by ϕ and let ω_λ be the part generated *in situ* while ω_δ is the part generated by diffusion. Thus:

$$\omega_\phi = \omega_\lambda + \omega_\delta \tag{3.3}$$

If the material is plastically incompressible all volume changes are elastic and thus by definition $\dot{\omega}_\phi = 0$. Thus:

$$\dot{\omega}_\phi = \dot{\omega}_\lambda + \dot{\omega}_\delta = 0 \tag{3.4}$$

In regard to ω_λ it is possible to envision this to consist of two parts, a reversible part ω_{λ_1}—which would have the physical meaning of recoverable volumetric changes, difficult of envision in inert granular media, but possible in metals and polymers— and an irreversible part ω_{λ_2}. The equations of evolution of ω_{λ_1} and ω_{λ_2} are:

$$\dot{\omega}_{\lambda_1} = B\dot{\phi} \tag{3.5}$$

$$\dot{\omega}_{\lambda_2} = -\dot{z}\,\partial k/\partial\phi \tag{3.6}$$

where B is a positive scalar. Equation (3.5) is self-explanatory and the negative sign accounts for the fact that ϕ is positive in compression while ω_{λ_1} is positive in dilation. Equation (3.6) is form-invariant with respect to its counterpart in local plasticity, where k is a function of p and z in which event: $\dot{\varepsilon}^p_{kk} = -\dot{z}\,\partial k/\partial p$. Thus:

$$\dot{\omega}_\lambda = -B\dot{\phi} - \dot{z}\,\partial k/\partial\phi \tag{3.7}$$

In regard to ω_δ, since this is generated by diffusion there must exist a "void flux" q_i such that:

$$\dot{\omega}_\delta = -q_{i,i} \tag{3.8}$$

Equation (3.4), the condition of elastic volumetric response, then leads to the following equation:

$$B\dot{\phi} + \dot{z}k_\phi = -q_{i,i} \tag{3.9}$$

If the material is rate-indifferent a gradient in ϕ will not, in itself, cause a void diffusion. What is needed is the gradient of the time rate of change of ϕ, i.e., $\nabla\dot{\phi}$. Quite clearly, q_i will be directed from regions of lower $\dot{\phi}$ toward regions of higher $\dot{\phi}$—i.e., from regions of lower void density changes, toward those of higher such—and thus:

$$q_i = a_{ij}\dot{\phi}_{,j} \tag{3.10}$$

where a_{ij} is the void diffusivity tensor which is positive definite. Equations (3.9) and (3.10) combine to give the following evolution eqn for $\dot{\phi}$:

$$B\dot{\phi}+\dot{z}k_{\phi} = -(a_{ij}\dot{\phi}_{,j})_{,i} \tag{3.11}$$

4. CONSTITUTIVE FORMULATION

We begin with eqn (4.1), i.e.,

$$f(\underset{\sim}{s}) = k(\phi, z) \tag{4.1}$$

Because ϕ is not a pressure, but a diffusion potential, the theory is associative. Thus:

$$\dot{e}^p = \dot{z}\underset{\sim}{f} \tag{4.2}$$

$$\dot{e}^p_{kk} = 0 \tag{4.2a}$$

The rate form of eqn (4.1)—the Prager consistency condition—is given by eqn (4.3):

$$\underset{\sim}{f}\cdot\dot{\underset{\sim}{s}} = k_{\phi}\dot{\phi}+k_z\dot{z} \tag{4.3}$$

At this point we use eqns (2.3a), (2.4), (4.2) and (4.3) to find the following relation for $\dot{\underset{\sim}{e}}$:

$$\dot{\underset{\sim}{e}} = \dot{\underset{\sim}{s}}/2\mu+(\underset{\sim}{f}/k_z)(\underset{\sim}{f}\cdot\dot{\underset{\sim}{s}}-k_{\phi}\dot{\phi}) \tag{4.4}$$

where a period between two tensors denotes their scalar product. Upon taking the inner product of both sides of eqn (4.4) with respect to $\underset{\sim}{f}$ one finds that:

$$H\dot{\underset{\sim}{s}}\cdot\underset{\sim}{f} = k_z\underset{\sim}{f}\cdot\dot{\underset{\sim}{e}}+\|\underset{\sim}{f}\|^2k_{\phi}\dot{\phi} \tag{4.5}$$

where H is given in eqn (4.6):

$$H = (k_z/2\mu)+\|\underset{\sim}{f}\|^2 \tag{4.6}$$

and $\|\cdot\|$ is the Euclidean norm. Thus, substituting for $\underset{\sim}{f}\cdot\dot{\underset{\sim}{s}}$ in eqn (4.4) the constitutive relation (4.7) is found between the deviatoric stress rate on one hand and the deviatoric strain rate and $\dot{\phi}$ on the other:

$$\dot{\underset{\sim}{s}} = 2\mu\dot{\underset{\sim}{e}}-(2\mu/H)\underset{\sim}{f}\otimes\underset{\sim}{f}\cdot\dot{\underset{\sim}{e}}+(\underset{\sim}{f}/H)k_{\phi}\dot{\phi} \tag{4.7}$$

where $\otimes$ is the outer product between two tensors. In a similar manner substitution for $\underset{\sim}{f}\cdot\dot{\underset{\sim}{s}}$ is eqn (4.3) gives the following equation for $\dot{z}$:

$$H\dot{z} = \underset{\sim}{f}\cdot\dot{\underset{\sim}{e}}-(k_{\phi}/2\mu)\dot{\phi} \tag{4.8}$$

The set of constitutive field equations

At this juncture the set of constitutive equations for the rates of (a) stress $\dot{\underset{\sim}{\sigma}}$ and (b) potential $\dot{\phi}$ are found by use of eqns (2.3b) and (4.7) for the former and eqns (3.11) and (4.8) for the latter. These are the following:

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{C}^{ep} \cdot \dot{\underset{\sim}{\varepsilon}} + H^{-1} k_{\phi} \underset{\sim}{f} \dot{\phi} \tag{4.9}$$

$$\dot{\phi} \left\{ -B + \frac{k_{\phi}^{2}}{2\mu H} \right\} = (a_{ij} \dot{\phi}_{,j})_{,i} + \frac{k_{\phi}}{H} f_{ij} \dot{\varepsilon}_{ij} \tag{4.10}$$

where $\underset{\sim}{C}^{ep}$, the elastoplastic tangent modulus, is now positive definite and symmetric, Valanis (1996), and is given in eqn (4.11):

$$C_{ijkl}^{ep} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{jk} - (2\mu/H) f_{ij} f_{kl} \tag{4.11}$$

We note that λ and μ are the elastic Lamé constants. These equations are supplemented by the equilibrium equation, given in rate form in eqn (4.12):

$$\dot{\sigma}_{ij,j} + \dot{f}_{i} = 0 \tag{4.12}$$

where f_i is the body force field.

Boundary conditions

Case (i). Stress boundary conditions. Permeable boundary. In the case $\dot{T}_i$ are prescribed on S, and in view of eqn (3.2), $\dot{\phi}$ is equal to $-\dot{T}_i n_i$. Since the boundary is permeable no further boundary condition is called for.

Case (ii). General boundary conditions. In this case traction rates $\dot{T}_i$ are prescribed on S_T and displacement rates $\dot{U}$ are prescribed on S_U such that $S_t U S_U = S$. Similarly, if S_{Φ} is a permeable boundary while S_Q is impermeable, such that $S_{\Phi} U S_Q = S$, then $\dot{\phi}$ is prescribed on S_{Φ}, in accordance with eqn (3.2), while on S_Q:

$$q_i n_i = a_{ij} \dot{\phi}_{,j} n_i = 0 \tag{4.13}$$

We note that case (i) is a special case of the general case (ii).

The boundary value problem

The boundary value problem is now defined as follows. Given the general boundary conditions (ii) on S and eqns (4.9), (4.10) and (4.12) in D to determine the velocity field $\dot{u}_i$ and the potential field rate $\dot{\phi}$ in D.

5. WELL-POSEDNESS

The boundary value problem is well-posed if a solution exists and is unique and stable.

Uniqueness and "stability" of the solution of the boundary value problem

Here we examine the type of "stability" the criterion for which is that: no solutions exist when the boundary conditions and the time rate of the body force field $\dot{f}_i$ are null. In linear problems this is also a criterion for uniqueness.

To demonstrate this, let $\{\dot{u}_i, \dot{\phi}\}_1$ be a solution set for the prescribed general boundary conditions, as in case (ii), and let $\{\dot{u}_i, \dot{\phi}\}_0$ be a solution when the boundary conditions are null. Because the boundary value problem is linear, then $\{\dot{u}_i, \dot{\phi}\}_1 + \{\dot{u}_i, \dot{\phi}\}_0$ would also be a solution. There would then exist a bifurcation and $(\text{solution})_1$ would not be "stable" since it could spontaneously "jump" to $(\text{solution})_1 + (\text{solution})_0$.

It now remains to show that the solution to the boundary value problem, as defined by eqns (4.9) and (4.10) and (4.12) and in the light of the prescribed boundary conditions of case (ii), is "stable". To this end, let Γ denote the following integral:

$$\Gamma = \int_S \dot{T}_i \dot{u}_i \, \mathrm{d}S + \int_S q_i n_i \dot{\phi} \, \mathrm{d}S + \int_V \dot{f}_i \dot{u}_i \, \mathrm{d}V \tag{5.1}$$

Upon analysis it then follows that Γ, in the light of eqn (4.9), (4.11) and (4.12), and upon use of the Green–Gauss theorem, is also given by eqn (5.2):

$$\Gamma = \int_V C^{ep}_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l} \, \mathrm{d}V + \int_V A \dot{\phi}^2 \, \mathrm{d}V + \int_V a_{ij} \dot{\phi}_{,i} \dot{\phi}_{,j} \, \mathrm{d}V \tag{5.2}$$

where $A = (k^2_\phi/2\mu H) - B$. We shall examine two different cases, (a) when $A \geqslant 0$ and (b) when $A < 0$.

Case (a): $A \geqslant 0$. We make note of the fact that C^{ep}_{ijkl} and a_{ij} are positive definite. Thus, in view of eqn (5.2) Γ is always positive for non null solutions. However when the boundary conditions are null and $\dot{f}_i = 0$, then Γ must be zero as a result of eqn (5.1). Since Γ cannot be positive and zero simultaneously, no non-null solutions can exist when the boundary conditions are null and $\dot{f}_i = 0$. Thus, if a solution exists it is "stable" in the sense that there is no bifurcation. This analysis, however, says nothing about existence.

To show uniqueness we note that if two solutions $\{\dot{u}_i, \dot{\phi}\}_1$ and $\{\dot{u}_i, \dot{\phi}\}_2$ exist then their difference would also be a solution but with null boundary and zero body-force field rate conditions. We have shown that because $\Gamma > 0$, this is not possible and thus the solution is unique.

Case (b): $A < 0$. In this case Γ is not always positive. If $\Gamma = 0$, then nothing can be said about "stability" or uniqueness. If $\Gamma < 0$, then again, since Γ cannot be simultaneously zero and negative, no solution can exist when the boundary condition are null. Thus, if a solution exists then it is "stable" in the above sense, and unique. In the following section, however, where we solve the problem of shearing of a material layer, we find that if $A < 0$ there are conditions whereby a solution does not exist. Under conditions where it does exist then it is periodic. Furthermore there are characteristic layer depths, for which the solution is not "stable" in the different sense that a small change in the boundary conditions produces a large change in the magnitude of the solution. This type of instability is tantamount to self-generation of energy.

6. SIMPLE SHEARING OF AN INFINITE LAYER

We complete the paper with the analysis of the following problem. A material layer of finite depth h and infinite length and width, is rigidly affixed to a rigid impermeable foundation at $y = 0$ (where y is the coordinate normal to the plane of the layer) and is subject to a uniform shear s ($s > 0$) and a uniform pressure p ($p \geqslant 0$) at $y = h$. We mention that this problem is geometrically equivalent to the pure torsion of a long thin tube.

The material is a Mises solid with isotropic hardening, in which case the configuration of the yield surface is given by eqn (6.1):

$$|s| = k(\phi, z) \tag{6.1}$$

the rate form of which is given by eqn (6.1a):

$$\dot{s}\,\mathrm{signum}(s) = k_\phi \dot{\phi} + k_z \dot{z} \tag{6.1a}$$

and since $s > 0$:

$$\dot{s} = k_\phi \dot{\phi} + k_z \dot{z} \tag{6.2}$$

Also:

$$\dot{e}^p = \dot{z}\,\partial|s|/\partial s = \dot{z}\,\text{signum}(s) = \dot{z} \tag{6.3}$$

In this particular one-dimensional problem eqn (3.9) becomes:

$$B\dot{\phi} - k_{,\phi}\dot{z} + \partial q/\partial y = 0 \tag{6.4}$$

while, in view of eqn (3.10):

$$q = a\,\partial\dot{\phi}/\partial y \tag{6.5}$$

where a (>0) is the diffusion coefficient. We now use eqns (6.2) and (6.5) and substitute for $\dot{z}$ and q in eqn (6.4) to find the following second-order ordinary differential equation for $\dot{\phi}$:

$$a\,\partial^2\dot{\phi}/\partial y^2 = A\dot{\phi} - (k_{\phi}/k_z)\dot{s} \tag{6.6}$$

where $A = k_{\phi}^2/k_z - B$. The boundary conditions are: $\partial\dot{\phi}/\partial y = 0$ at $y = 0$; $\dot{\phi} = \dot{p}$ at $y = h$.

Case of A > 0. We introduce the parameter b where: $b = (A/a)^{1/2}$. In the light of the boundary conditions the solution to eqn (6.6) is then found to be:

$$\dot{\phi} = \kappa_{\phi}^{-1}\dot{s}\left(1 - \frac{\cosh by}{\cosh bh}\right) + \dot{p}\frac{\cosh by}{\cosh bh} \tag{6.7}$$

Of particular interest is the case where $\dot{p}$ is zero. For metals it represents simple torsion of a thin tube at constant (or zero) pressure at the loading end. In this case:

$$k_{\phi}\dot{\phi} = \dot{s}\left(1 - \frac{\cosh by}{\cosh bh}\right) \tag{6.8}$$

Remark. In view of eqn (6.2) plastic action will take place (for $k_z > 0$) whenever:

$$\dot{s} > k_{\phi}\dot{\phi} \tag{6.9}$$

Because the term multiplying $\dot{s}$ on the right hand side of eqn (6.9) is always less than unity in the interval $0 \leqslant y \leqslant h$, condition (6.10) is satisfied for all y. Thus, plastic action will take place in the entire domain of the layer whenever $\dot{s} > 0$.

Furthermore use of eqn (6.2) gives:

$$k_z\dot{z} = \dot{s}\frac{\cosh by}{\cosh bh} \tag{6.10}$$

Or, in view of eqn (6.3):

$$k_z\dot{e}^p = \dot{s}\frac{\cosh by}{\cosh bh} \tag{6.11}$$

Now let:

$$k(\phi, z) = \kappa(\phi)(1 - e^{-\alpha z}) + s_0 \tag{6.12}$$

where s_0 is the yield stress in shear, in light of eqn (6.1), $1 - e^{\alpha z}$ is a common form of the

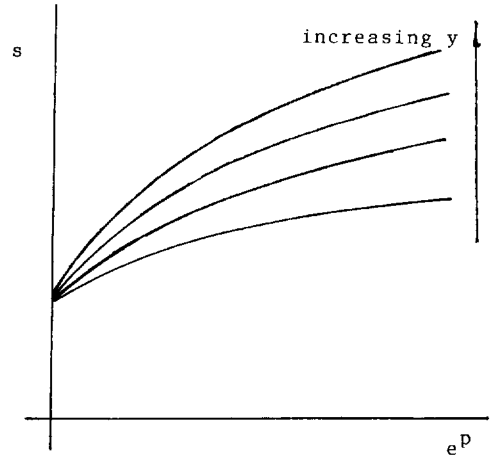

Fig. 1. Shear stress versus plastic shear strain at increasing distance from bottom rigid impermeable surface.

hardening function and $k(\phi)$ depicts the dependence of the yield surface on ϕ (not p as in standard plasticity). Then:

$$b = 2\,\mathrm{d}\kappa/\mathrm{d}\phi(a\alpha\kappa)^{-1/2}\cosh(\alpha/2)z \tag{6.13}$$

$$k_z = \alpha\kappa(\phi)\,e^{-\alpha z} \tag{6.14}$$

Equation (6.13) may be integrated to give s as a function of z—and e^p since in view of eqn (6.3) $e^p = z$. Thus:

$$s = s_o + \int_o^z \alpha\kappa\,e^{-\alpha z}\frac{\cosh bh}{\cosh by}dz \tag{6.15}$$

A schematic representation of the stress–plastic strain curves at various depths is given in Fig. 1. Evidently the material is softest at the top of the layer and hardens monotonically as the depth increases, becoming hardest at the bottom. The strain rate distribution is not uniform—as depicted by the local theory—and the plastic strain rate decreases with depth at fixed $\dot{s}$.

Pressure sensitivity of the plastic strain rate

Also of interest is the variation in plastic shear strain, in a case encountered physically, where the pressure varies at constant applied stress. Then, it follows from eqn (6.7):

$$\dot{\phi} = \dot{p}\frac{\cosh by}{\cosh bh} \tag{6.16}$$

Thus, $\dot{\phi}$ is a maximum at $y = h$, the surface of applied pressure, and smallest at $y = 0$. We now recall eqns (6.2) and (6.16) to find:

$$\dot{z} = -(k_\phi/k_z)\dot{p}\frac{\cosh by}{\cosh bh} \tag{6.17}$$

If $\dot{p}$ is positive then $\dot{z}$ is negative and plastic deformation cannot take place. Thus $\dot{e}^p = 0$. On the other hand if $\dot{p}$ is negative, then $\dot{z}$ is positive and thus:

$$\dot{e}^p = -(k_\phi/k_z)\dot{\phi} = (k_\phi/k_z)|\dot{p}|\frac{\cosh by}{\cosh bh} \tag{6.18}$$

In this case the plastic strain increases with decrease of pressure, as would be expected physically, but at a rate which is a maximum at the upper boundary because there $\dot{\phi}$ is a maximum.

This result is in broad agreement with observations of unstable motion of top layers of soils under pressure.

7. SOFTENING AND PATTERNED DEFORMATION

Case of $A < 0$

In this last section we shall show that softening alone i.e., in the absence of variations in pressure, in fact under conditions of $p \equiv 0$, gives rise to solutions that are subject to instability and (ii) leads to geometrically patterned deformation. To show this we set $k_z < 0$, in which event $A < 0$ and choose k to be the softening function depicted in eqn (7.1):

$$k(\phi, z) = (k_o + \beta\phi)\, e^{-\alpha z} + c_o \tag{7.1}$$

where k_o, β, α and c_o are all positive material constants. Now,

$$k_\phi = \beta\, e^{-\alpha z} \tag{7.2}$$

Furthermore at $p \equiv 0$,

$$k_z = -k_o \alpha\, e^{-\alpha z} \tag{7.3}$$

Thus setting $B = 0$, for ease of illustration,

$$b^2 = -b_o^2\, e^{-\alpha z} \tag{7.4}$$

where $b_o^2 = \beta^2/a\alpha k_o$. Evidently b^2 is negative and b is imaginary and equal to ib^* where b^* is equal to $b_o e^{-\alpha z/2}$ and is always real.

Substituting this value of b in solution for $\dot{e}^p$ given by eqn (6.12) we find a new periodic form of the solution given in eqn (7.5):

$$-|k_z|\dot{e}^p = \dot{s}\frac{\cos b^* y}{\cos b^* h} \tag{7.5}$$

Discussion

We remark on three significant differences between the new "softening" solution and its previous "hardening" counterpart i.e., eqn (6.12). (i) The sign of $\dot{e}^p$ is opposite to that of $\dot{s}$ as it must be, given the fact that $\dot{s}$ is constant in the layer; (ii) $\dot{e}^p$ is of periodic character and thus gives rise to patterned deformation in the form of periodic bands; (iii) a solution does not exist when $b^*h = \pi/2$, at which point $\cos b^*h$ vanishes and $\dot{e}^p$ becomes infinite. The solution becomes "unstable" when $b^*h = \pi/2 - \delta$ where δ is a vanishingly small number. Then:

$$-|k_z|\dot{e}^p = \dot{s}\frac{\cos b^* y}{\delta} \tag{7.6}$$

in which event $\dot{e}^p$ becomes large at very small values of $\dot{s}$. Of interest is the fact that a solution exists, it is periodic and unique, except for values of $\delta = 0$.

8. EXISTENCE AND UNIQUENESS OF SOLUTION TO THE BOUNDARY VALUE PROBLEM

In Section 5 we showed stability and uniqueness of the solution to the boundary value problem in the event that a solution exists. In this section we discuss the question of existence and provide appropriate proof. We limit the discussion to variational solutions.

We recall the pertinent equations in D, i.e.,

$$\dot{\sigma}_{ij} = C^{ep}_{ijkl}\dot{\varepsilon}_{kl} + F_{ij}\dot{\phi} \tag{8.1}$$

$$A\dot{\phi} = (a_{ij}\dot{\phi}_{,j})_{,i} + F_{ij}\dot{\varepsilon}_{ij} \tag{8.2}$$

$$\dot{\sigma}_{ij,j} + \dot{f}_i = 0 \tag{8.3}$$

where $F_{ij} - H^{-1}k_{\phi}f_{ij}$, and the pertinent boundary conditions on S, i.e.,

$$\dot{T}_i = \dot{\sigma}_{ij}n_j; \quad \dot{\phi} = \dot{T}_i n_i \tag{8.4a,b}$$

The "solution" to the boundary value problem then consists in finding functions $\dot{u}_i$ and the $\dot{\phi}$, continuous and differentiable in D that satisfy eqns (8.1–8.3) in D and the boundary conditions (8.4a,b) on S.

The discussion will be limited to variational solutions. See Courant and Hilbert (1962). We multiply both sides of eqn (8.3) by $\delta\dot{u}_i$, where δ is the variation operator and integrate over D. Upon use of eqn (8.1) and the Green–Gauss theorem we find the following variational form of these equations:

$$\int_V C^{ep}_{ijkl}\dot{u}_{i,j}\delta\dot{u}_{k,l}\,\mathrm{d}V + \int_V \dot{\phi}F_{ij}\delta\dot{u}_{i,j}\,\mathrm{d}V = \int_S \dot{T}_i\delta\dot{u}_i\,\mathrm{d}S \tag{8.5}$$

We now multiply both sides of eqn (8.2) by $\delta\dot{\phi}$, integrate over D and use the Green–Gauss theorem to find the following variational form of eqn (8.2):

$$\int_V A\dot{\phi}\delta\dot{\phi}\,\mathrm{d}V + \int_V a_{ij}\dot{\phi}_{,i}\delta\dot{\phi}_{,j}\,\mathrm{d}V - \int F_{ij}\dot{u}_{i,j}\delta\dot{\phi} = -\int a(\delta\dot{\phi}/\partial n)\delta\dot{\phi} \tag{8.6}$$

Without going into excessive discussion, we note that eqns (8.5) and (8.6) would provide a variational solution to the problem if $\delta\dot{\phi}/\partial n$ were known on S. However this is not the case here since it is $\dot{\phi}$ that is known on S.

To deal with this specific case we set:

$$\phi = \phi^* + \phi' \tag{8.7}$$

where ϕ^* is any (known) function that satisfies the boundary conditions, i.e.,

$$\phi^* = \phi \quad \text{on } S \tag{8.8}$$

and ϕ' is such that:

$$\phi' = 0 \quad \text{on } S \tag{8.9}$$

we note that $\delta\phi = \delta\phi'$, since ϕ^* is fixed and that:

$$\delta\phi' = 0 \quad \text{on } S \tag{8.10}$$

Upon substitution of eqn (8.7) into eqn (8.5) and (8.6) we arrive at the following variational formulation in terms of $\dot{\phi}'$:

$$\int_V C^{ep}_{ijkl}\dot{u}_{i,j}\delta\dot{u}_{k,l}\,\mathrm{d}V + \int_V \dot{\phi}' F_{ij}\delta\dot{u}_{i,j} = \int_S \dot{T}_i\delta\dot{u}_i - \delta F^* \tag{8.11}$$

$$\int_V A\dot{\phi}'\delta\dot{\phi}'\,\mathrm{d}V + \int_V a_{ij}\dot{\phi}'_{,i}\delta\dot{\phi}'_{,j}\,\mathrm{d}V - \int F_{ij}\dot{u}_{i,j}\delta\dot{\phi}' = -\delta A^* \tag{8.12}$$

with the following definitions:

$$F^* = \int_V F_{ij}\dot{\phi}^*\dot{u}_{i,j}\,\mathrm{d}V \tag{8.13a}$$

$$A^* = \int_V A\dot{\phi}^*\dot{\phi}'\,\mathrm{d}V + \int_V a_{ij}\dot{\phi}^*_{,j}\dot{\phi}_{,i} \tag{8.13b}$$

Note should be made of the fact that the surface integral on the right hand side of eqn (8.6) is zero since $\delta\dot{\phi}'$ vanishes on S.

At this point we use the usual methods of variational calculus to obtain the solution to eqns (8.11) and (8.12). We thus let ϕ^r be a set of continuous and differentiable functions that vanish on S and span ϕ' in the sense that:

$$\dot{\phi}' = \sum_1^m \phi^r\Phi_r + \delta_m \tag{8.14}$$

such that the "error" $\delta_m \to 0$ and $m \to \infty$. In the same vein let:

$$\dot{u}_i = \sum_1^n u_i^r U_r - \delta^*_{ni} \tag{8.15}$$

where $\delta^*_{ni} \to 0$ as $n \to \infty$, and U_r are generalised "coordinates" without directionality. Then to within the indicated error and for any finite n:

$$\delta\dot{\phi}' = \Sigma\phi^r\delta\Phi_r; \quad \delta\dot{u}_i = u_i^r\delta U_r \tag{8.16a,b}$$

The solution of eqns (8.11) and (8.12) is then reduced to determining the coefficients U_r and Φ_r.

Upon substitution of eqn (8.16a,b) into eqns (8.11) and (8.12) and use of the standard variational arguments one obtains the following set of simultaneous linear algebraic equations in the unknowns ϕ^r and U_r:

$$C^{rs}U_s + F^{rs}\phi_s = T^r - F^{*s} \tag{8.17}$$

$$-F^{rs}U_s + A^{rs}\phi_s = -A^{*r} \tag{8.18}$$

with the following definitions:

$$T^r = \int_S \dot{T}_i u_i^r\,\mathrm{d}S \tag{8.19}$$

$$F^{*r} = \int_V F_{ij}\dot{\phi}u^r_{i,j}\,\mathrm{d}V \tag{8.20}$$

$$A^{*r} = \int_V \{A\phi^*\phi^r + a_{ij}\phi^*_{,i}\phi^*_{,j}\}\,\mathrm{d}V \tag{8.21}$$

$$C^{rs} = \int_V C^{ep}_{ijkl}u^r_{i,j}u^s_{k,l}\,\mathrm{d}V \tag{8.22}$$

$$F^{rs} = \int_V \phi^r F_{ij}u^s_{i,j} \tag{8.23}$$

$$A^{rs} = \int_V \{A\phi^r\phi^s + a_{ij}\phi^r_{,i}\phi^s_{,j}\}\,\mathrm{d}V \tag{8.24}$$

A perusal of eqns (8.17) and (8.18) shows that a solution to the set these simultaneous linear algebraic equations exists and is unique if the matrices C^{rs} and A^{rs} are non-singular. Now C^{rs} is symmetric and positive definite because C^{ep}_{ijkl} is such. The diffusion matrix a_{ij} is symmetric and positive definite. Thus if A is non-negative then A^{rs} is positive definite as well. In this event C^{rs} and A^{rs} are both non-singular and thus a variational solution to the boundary value problem exists and is unique.

Remark. The sign of A. We recall that:

$$A = (k^2_\phi/2\mu H) > B \tag{8.25}$$

If during an experiment a material does not exhibit banding or patterned deformation (not all deformation fields exhibit banding), then we infer that A is positive, i.e., $(k^2_\phi/2\mu H)B$, and the boundary value problem is well posed. If A is negative A^{rs} may also be negative (or zero). In the case of simple shearing patterned deformation results when $A < 0$. The general problem is more complex and will be addressed in future work.

9. CONSTITUTIVE FORMULATION IN THE PRESENCE OF ISOTROPIC AND KINEMATIC HARDENING

Kinematic hardening is introduced through the back stress tensor $\underset{\sim}{\alpha}$. Equation (2.1) now becomes:

$$f(\underset{\sim}{s}-\underset{\sim}{\alpha}) = k(\phi, z) \tag{9.1}$$

As before:

$$d\underset{\sim}{e}^p = \underset{\sim}{f}\,\mathrm{d}z \tag{9.2}$$

and therefore:

$$\|d\underset{\sim}{e}\| = \|\underset{\sim}{f}\|\,\mathrm{d}z \tag{9.3}$$

The Prager consistency condition, in the light of eqn (9.1) now becomes:

$$\underset{\sim}{f}\cdot(d\underset{\sim}{s}-d\underset{\sim}{\alpha}) = \mathrm{d}k \tag{9.4}$$

It was shown previously by Valanis (1980), in connection with his work on endocrinic plasticity, that in the presence of plastic incompressibility, $\underset{\sim}{\alpha}$ is a linear functional of the plastic strain history, with respect to a memory kernel $\rho_1(z)$, i.e.,

$$\underset{\sim}{\alpha} = \int_o^z \rho_1 (z - z' \mathrm{d}\underset{\sim}{e}^P \tag{9.5}$$

and thus:

$$d\underset{\sim}{\alpha} = \rho_1(0)\,\mathrm{d}\underset{\sim}{e}^P + \underset{\sim}{h}\,\mathrm{d}z \tag{9.6}$$

where:

$$\underset{\sim}{h} = \int_o^z \rho_1'(z - z')\,\mathrm{d}e^P \tag{9.7}$$

where $\rho_1' \equiv \mathrm{d}\rho_1/\mathrm{d}z$.

Use of eqn (9.6) and upon substitution for $\mathrm{d}\alpha$ in eqn (9.4) one obtains the following result:

$$\underset{\sim}{f}\cdot\dot{\underset{\sim}{s}} = (H_s + k_z)\dot{z} + k_\phi\dot{\phi} \tag{9.8}$$

where:

$$H_s = \rho_1(0) + (\underset{\sim}{f}\cdot\underset{\sim}{h})/\|\underset{\sim}{f}\|^2 \tag{9.9}$$

Comparison of eqn (9.8) with eqn (4.3) shows that the only change, relative to isotropic hardening, is the augmentation of the isotropic hardening coefficient k_z by the kinematic hardening coefficient H_s, otherwise the form of the equations remains precisely the same. Thus the hardening function H, previously shown in eqn (4.6), is now given by eqn (9.10):

$$H = (H_s + k_z)/2\mu + \|\underset{\sim}{f}\|^2 \tag{9.10}$$

We remark that H_s is non-negative but leave the proof to the reader. Thus, H is positive if $k_z > 0$.

Furthermore, the constitutive field eqns (4.9) and (4.10) remain precisely the same except that H is now given by eqn (9.10). The constitutive formulation in the presence of isotropic as well as kinematic hardening is now complete.

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